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HW6 December 1, 2008

1. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on X, show that $(X, \mathcal{T}_1 \cap \mathcal{T}_2)$ is also a topological space. Give an example where $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X.

   Solution. In order to show that $(X, \mathcal{T}_1 \cap \mathcal{T}_2)$ is a topological space, I need to show: (i) $\mathcal{T}_1 \cap \mathcal{T}_2$ contains both $\emptyset$ and X, (ii) $U, V \in \mathcal{T}_1 \cap \mathcal{T}_2 \implies U \cap V \in \mathcal{T}_1 \cap \mathcal{T}_2$ and (iii) $\{U_\alpha | \alpha \in A\} \in \mathcal{T}_1 \cap \mathcal{T}_2$, A is some indexing set, $\implies \bigcup_\alpha U_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$.

   (i) Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, $\emptyset \in \mathcal{T}_1$ and $\emptyset \in \mathcal{T}_2$. Therefore, $\emptyset \in \mathcal{T}_1 \cap \mathcal{T}_2$. The same argument follows for X, so $X \in \mathcal{T}_1 \cap \mathcal{T}_2$.

   (ii) Pick some $U, V \in \mathcal{T}_1 \cap \mathcal{T}_2$. Since $\mathcal{T}_1$ and $\mathcal{T}_2$ are topologies on X, $U \cap V \in \mathcal{T}_1$ and $U \cap V \in \mathcal{T}_2$. Therefore, $U \cap V \in \mathcal{T}_1 \cap \mathcal{T}_2$.

   (iii) Pick some collection of subsets $\{U_\alpha | \alpha \in A\} \in \mathcal{T}_1 \cap \mathcal{T}_2$. So, $\{U_\alpha | \alpha \in A\} \in \mathcal{T}_1$ and $\{U_\alpha | \alpha \in A\} \in \mathcal{T}_2$. It follows that $\bigcup_\alpha U_\alpha \in \mathcal{T}_1$ and $\bigcup_\alpha U_\alpha \in \mathcal{T}_2$. Therefore, $\bigcup_\alpha U_\alpha \in \mathcal{T}_1 \cap \mathcal{T}_2$.

   $\therefore$ $(X, \mathcal{T}_1 \cap \mathcal{T}_2)$ is a topological space. Note: all of the arguments work if any set U, V, or one of the U_α's is the empty set.

   Now, I need to find a counterexample such that $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X. One such example that is very simple is when X is the three-point set $\{1, 2, 3\}$. Define two topologies on X s.t.

   $$\begin{aligned} \mathcal{T}_1 &= \{\{1,3\}, \{2,3\}, \{3\}, X, \emptyset\} \\ \mathcal{T}_2 &= \{\{1,2\}, \{2,3\}, \{2\}, X, \emptyset\}. \end{aligned}$$

   It is easy to show that the above sets are topologies on X, since (i) they both contain $\emptyset$ and X, (ii) any intersection of two elements of $\mathcal{T}_i$ is in $\mathcal{T}_i$ $(i = 1, 2)$ and (iii) a union of a collection of sets in $\mathcal{T}_i$ is in $\mathcal{T}_i$. Take the union to get

   $$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2\}, \{3\}, X, \emptyset\}.$$

   Take $U = \{1, 2\}$ and $V = \{1, 3\}$. Clearly, $U, V \in \mathcal{T}_1 \cup \mathcal{T}_2$, however $U \cap V = \{1\} \notin \mathcal{T}_1 \cup \mathcal{T}_2$. Therefore, property (ii) of topologies is failed and $\mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X.

2. **The structure of open sets on $\mathbb{R}$.**

   In what follows, we are considering the standard (metric) topology on $\mathbb{R}$.

   (a) Let S be a nonempty open subset of $\mathbb{R}$. For each $x \in S$, let $A_x = \{a \in \mathbb{R} : (a, x] \subseteq S\}$ and $B_x = \{b \in \mathbb{R} : [x, b) \subseteq S\}$. Show that A_x and B_x are both non-empty.

Solution. Although, I didn't say this in the problem, we need to insist that the interval $(a, x]$ in the definition of A_x is non-empty and likewise $[x, b)$ is non-empty. This implies in particular, $a \in A_x \implies a < x$ and $b \in B_x \implies b > x$. So, $x \notin A_x$ and $x \notin B_x$, and we need a more refined argument to show that $A_x \neq \emptyset$ and likewise for B_x.

Since S is an open subset of $\mathbb{R}$, so for all $x \in S$ there exists ϵ such that $B(x, \epsilon) \subseteq S$. This means that there exists ϵ such that $(x - \epsilon, x + \epsilon) \subseteq S$. Since $(a, x] \subseteq (x - \epsilon, x + \epsilon)$, there must exist $a = x - \epsilon$ such that $(a, x] \subseteq S$. So, A_x is non-empty.

Make the same argument for B_x non-empty.

(b) Where $x \in S$ as above, if A_x is bounded below, let $a_x = \inf(A_x)$. Otherwise, let $a_x = -\infty$, and define b_x in a corresponding manner. Show that $x \in I_x = (a_x, b_x) \subseteq S$.

Solution. $(a, x] \subseteq (a_x, x]$ for all $a \in A_x$ since a_x is a greatest lower bound for A_x (or $-\infty$) and so $y \in (a, x]$ implies $y \in (a_x, x]$. Likewise, $[x, b) \subseteq [x, b_x)$ for all $b \in B_x$ since b_x is a least upper bound for B_x (or ∞) and so $y \in [x, b)$ implies $y \in [x, b_x)$. It follows that $x \in I_x$ since $x \in (a, x] \cup [x, b) \subseteq (a_x, x] \cup [x, b_x) = (a_x, b_x)$.

Suppose $y \in I_x$. If $y \in (a_x, x]$ then $y \in (a, x]$ for some $a \in A_x$. Specifically, if $a_x = -\infty$ choose $a = y - 1$, otherwise choose $a = \frac{a_x + y}{2}$. Likewise, if $y \in [x, b_x)$ then $y \in [x, b)$ for some $b \in B_x$. Specifically, if $b_x = \infty$ choose $b = y + 1$, otherwise choose $b = \frac{b_x + y}{2}$. $(a, x] \subseteq S$ for all $a \in A_x$ and $[x, b) \subseteq S$ for all $b \in B_x$, so $y \in S$.

$therefore x \in I_x \subseteq S$.

(c) Show that $S = \bigcup_x I_x$.

Solution. From the previous problem, we know that $I_x \subseteq S$ for all $x \imath S$. It follows that $\bigcup_x I_x \subseteq S$.

Suppose $y \in S$. Then $y \in I_y \subseteq \bigcup_x I_x$, so $S \subset \bigcup_x I_x$.

$\therefore S = \bigcup_x I_x$.

(d) Show that the intervals I_x give a partition of S, *i.e.* for $x, y \in S$, either $I_x = I_y$ or $I_x \cap I_y = \emptyset$.

Solution. Suppose $x, y \in S$ and $y \in I_x$. Then $y \in (a_x, x]$ or $y \in [x, b_x)$, but we can assume without loss of generality that $y \in (a_x, x]$ (otherwise, just relabel x and y but keep $y \in I_x$).

Suppose $z \in (a_x, x]$. But $(a_x, y] \subseteq (a_x, y] \cup (y, x] = (a_x, x] \subseteq S$ since $(a_x, x] \subseteq I_x \subseteq S$ so $a_x \in A_y$. This implies $z \in (a_y, x]$ since a_y is the greatest lower bound for A_y. Suppose $z \in (a_y, x]$. But $(a_y, x] = (a_y, x] \cup (y, x] \subseteq S$ since $(a_y, y] \subseteq I_y \subseteq S$ and $(y, x] \subseteq I_x \subseteq S$ so $a_y \in A_x$. This implies $z \in (a_x, x]$ since a_x is the greatest lower bound for A_x. Thus, $(a_x, x] = (a_y, x]$ so $a_x = a_y$.

Suppose $z \in [x, b_x)$. but $[x, b_x) \subseteq [y, x) \cup [x, b_x) = [y, b_x) \subseteq S$ since $[y, x) \subseteq I_x \subseteq S$ and $[x, b_x) \subseteq I_x \subseteq S$ so $b_x \in B_y$. This implies $z \in [x, b_y)$ since b_y is the least upper bound for B_y. Suppose $z \in [x, b_y)$. But $[x, b_y) \subseteq [y, x) \cup (x, b_y) = [y, b_y) \subseteq S$ since

$[y, b_y) \subseteq I_y \subseteq S$ so $b_y \in B_x$. This implies $z \in [x, b_x)$ since b_x is the least upper bound for B_x. Thus, $[x, b_x) = [x, b_y)$ so $b_x = b_y$.

It follows that $I_x = I_y$.

Suppose $x, y \in S$ and $y \notin I_x$. If $I_x \cap I_y \neq \emptyset$, then either $I_x \cap I_y = (a_x, b_y)$, in which case $(a_x, x] \subseteq (a_y, x] \subseteq S$ and so $a_y \in A_x$, or $I_x \cap I_y = (a_y, b_x)$, in which case $[x, b_x) \subseteq [x, b_y) \subseteq S$ and so $b_y \in B_x$. In this case, either a_x is not a greatest lower bound for A_x or b_x is not a least upper bound for B_x, which is a contradiction. It follows that $I_x \cap I_y = \emptyset$.

$\therefore$ either $I_x = I_y$ or $I_x \cap I_y = \emptyset$.

(e) Show that the set of distinct intervals $\{I_x : x \in S\}$ is countable.

Solution. Since the rational numbers $\mathbb{Q}$ are dense in $\mathbb{R}$, each interval I_x must contain a rational point. Therefore, since the rational numbers are countable, the set of distinct intervals is countable.

$\therefore$ the set of distinct intervals $\{I_x : x \in S\}$ is countable.

(f) Prove proposition 3.1.4, Pg. I-112 in Dr. Flaschka's notes.

Proposition. *Every (open) set, other than $\emptyset$, in the metric topology of $\mathbb{R}$ is a countable union of disjoint intervals.*

Proof. Let U, not $\emptyset$, be an element of the metric topology of $\mathbb{R}$, and $x \in U$. There exists a largest open interval (α, β) that contains x. The last statement is true for bounded intervals by the supremum property on $\mathbb{R}$; it is also true for unbounded intervals (take all of $\mathbb{R}$, for example).

I will take advantage of the fact that the rational numbers $\mathbb{Q}$ are countable, $q_1, q_2, \ldots$. Follow through all of these until one rational number that is in U, call it q_m is found. Now, construct (a_m, b_m) to be the largest interval in U that contains q_m. Following the above step for multiple m, there will be many duplications in the list (a_m, b_m), but at most countably many distinct ones, since we are working with countable elements q_m. Any union of two of these (a_m, b_m) must be disjoint, since the union would be larger than the constructed largest interval.

Let $V = \bigcup_m (a_m, b_m)$. For all m, $(a_m, b_m) \in U$, therefore $V \subseteq U$. For the other direction, if $x \in U$, $x \in (\alpha, \beta) \in U$, for some α, β. Since the rationals are dense in $\mathbb{R}$, $\exists q_m \in (\alpha, \beta)$. This means that the longest interval containing q_m from above must be longer than (α, β). Taking the union over all such intervals containing q_m in U shows $U \subseteq V$. Therefore, $U = V$.

Therefore, every (open) set, other than $\emptyset$, in the metric topology of $\mathbb{R}$ is a countable union of disjoint intervals.

Alternate proof

Proof. By Problem 6.2(c), if S is any non-empty open set in the metric topology on $\mathbb{R}$, then $S = \bigcup_x I_x$. By Problem 6.2(e), the set of distinct intervals $\{I_x : x \in S\}$ is countable.

$\therefore$ Every (open) set other than $\emptyset$, in the metric topology on $\mathbb{R}$ is a countable union of distinct intervals.

3. $(X, \mathcal{T})$ and $(Y, \mathcal{S})$

(a) If $W \subseteq X$ is a subset, show that the *relative* topology of W defined by $O \subseteq W$ is open only if $O = W \cap U$ for some $U \in \mathcal{T}$ is indeed a topology.

Solution. Redefine the proposed topology more symbolically as

$$\mathcal{T}_W = \{W \cap U | U \in \mathcal{T}\}.$$

As before, I need to show the three properties of a topology hold for T_W.

(i) Since $\mathcal{T}$ is a topology, take $U = \emptyset \in \mathcal{T}$, $W \cap \emptyset = \emptyset \in \mathcal{T}_W$. Now, take $U = X \in \mathcal{T}$, $W \cap X = W$ since $W \subseteq X$. Therefore, $W \in \mathcal{T}_W$.

(ii) Pick $U_W, V_W \in \mathcal{T}_W$. Therefore, $\exists U_X, W_X \in \mathcal{T}$ s.t. $U_W = W \cap U_X$ and $V_W = W \cap V_X$. Look at the intersection of U_W, V_W

$$\begin{aligned} U_W \cap V_W &= (W \cap U_X) \cap (W \cap V_X) \\ &= W \cap (U_X \cap V_X). \end{aligned}$$

Since, $U_X, V_X \in \mathcal{T}$ and $\mathcal{T}$ is a topology on X, $U_X \cap V_X \in \mathcal{T}$. Therefore, $U_W \cap V_W \in T_W$.

(iii) Pick $\{U_\alpha | \alpha \in A\} \in \mathcal{T}_W$. Therefore, $\exists V_\alpha \in \mathcal{T}$ s.t. $U_\alpha = W \cap V_\alpha$. Take the union of these sets

$$\begin{aligned} \bigcup_\alpha U_\alpha &= \bigcup_\alpha W \cap V_\alpha \\ &= W \cap \bigcup_\alpha V_\alpha. \end{aligned}$$

But, since $\mathcal{T}$ is a topology, $\bigcup_\alpha V_\alpha \in \mathcal{T}$. Therefore $\bigcup_\alpha U_\alpha \in \mathcal{T}_W$.

$\therefore$ $\mathcal{T}_W$ is a topology since all properties of topologies hold.

(b) If (X, d) is a metric space, and $W \subseteq X$, then we have shown in an earlier homework that the restriction d to W gives a metric space. We can define two topologies on W, the relative topology on W from the metric topology on X, and the metric topology on W induced by the restriction of d onto W. Show that the two topologies are the same.

Solution. Let $\mathcal{T}$ be the topology on X induced by d, $\mathcal{T}_r$ be the relative topology on W from the metric topology on X, and the metric topology on W induced by the restriction of the metric d to W.

We want to show that $\mathcal{T}_r = \mathcal{T}_d$.

$\subseteq$: We want to use the characterization in part 1a) to show that $\mathcal{T}_r$ is finer than $\mathcal{T}_d$. Let $w \in W$ be an arbitrary point, and let $O \in \mathcal{T}_r$ contain w; we want to find a $V \in \mathcal{T}_d$ that contains w such that $V \subseteq O$. By our hypothesis $O = W \cap U$ for some $U \in \mathcal{T}$ open in the topology for X. But U is generated by open balls $B(x, r)$ in the d_X metric (with $x \in X$, not necessarily in W), so we can write

$$O = W \bigcap \left(\bigcup_\alpha B^\alpha_{d_X}(x, r) \right) = \bigcup_\alpha \left(W \cap B^\alpha_{d_X}(x, r) \right).$$

(Here the α superscript is meant to be responsible for assigning the values x, r). For every intersection $W \cap B_{d_X}(x, r)$ that contains w, we can find a corresponding $B_{d_W}(\tilde{w}, \tilde{r}) \in \mathcal{T}_r$. We know that there must be some intersections that contain w, since O contains w Taking an arbitrary union of these sets $B_{d_W}(\tilde{w}, \tilde{r})$ will also be in $\mathcal{T}_d$, and let us call this union V. Then

$$\begin{aligned} V &= \bigcup_\alpha B^\alpha_{d_W}(\tilde{w}, \tilde{r}) \\ &\subseteq W \cap \left(\bigcup_\alpha B^\alpha_{d_X}(x, r) \right) \\ &= \bigcup_\alpha \left(W \cap B^\alpha_{d_X}(x, r) \right) \\ &= O \end{aligned}$$

This gives us $V \subseteq O$, with $V \in \mathcal{T}_d$ an open set containing w. Therefore $\mathcal{T}_r$ is finer than $\mathcal{T}_d$.

$\supseteq$ For the opposite containment, suppose $O \in \mathcal{T}_d$ is an arbitrary open set. We first note $B_{d_W}(w, r) = B_{d_X}(w, r)|_W = W \cap B_{d_X}(w, r)$, $W \cap B_{d_X}(w, r)$, where $B_{d_X}(w, r) \in \mathcal{T}$. Now $O \in \mathcal{T}_d$ can be written as $\bigcup_\alpha B^\alpha_{d_W}(w, r)$, so we have

$$O = \bigcup_\alpha B^\alpha_{d_W}(w, r) = W \cap \left(\bigcup_\alpha B^\alpha_{d_X}(w, r) \right)$$

Since $\bigcup_\alpha B^\alpha_{d_X}(x, r) \in \mathcal{T}$, then $O \in \mathcal{T}_r$.

$\therefore \quad \mathcal{T}(d)|_W$ and $\mathcal{T}(d|_W)$ are equivalent.

(c) For every $(x, y) \in X \times Y$, define the collection of neighborhoods by

$$\mathcal{N}(x, y) = \left\{ U_x \times V_y | x \in U_x \in \mathcal{T},\ y \in V_y \in \mathcal{S} \right\}$$

Show that this collection of neighborhoods defines a *local base*. The topology generated by this local base is called the *product topology*.

Solution. To show that $\mathcal{N}(x,y)$ is a local base, I need to show 3 properties, (i) $(U_x, V_y) \in \mathcal{N}(x,y) \implies (x,y) \in U_x \times V_y$ and $(X,Y) \in \mathcal{N}(x,y)$, (ii) $U_1 \times V_1, U_2 \times V_2 \in \mathcal{N}(x,y) \implies \exists U_3 \times V_3$ s.t. $U_3 \times V_3 \subseteq (U_1 \cap U_2) \times (V_1 \cap V_2)$, and (iii') $U_1 \times V_1 \in \mathcal{N}(x,y)$ and $(x_1, y_1) \in U_1 \times V_1 \implies \exists U_2 \times V_2 \in \mathcal{N}(x_1,y_1)$ s.t. $U_2 \times V_2 \subseteq U_1 \times V_1$.

(i) is easy to show. Given $U_1 \times V_1 \in \mathcal{N}(x,y)$, $(x,y) \in U_1 \times V_1$ by definition. Also, since $U_1 \times V_1 \subseteq X \times Y$, $(x,y) \in X \times Y \implies X \times Y \in \mathcal{N}(x,y)$.

(ii) Given $U_1 \times V_1$, $U_2 \times V_2 \in \mathcal{N}(x,y)$. Take the intersection,

$$(U_1 \times V_1) \cap (U_2 \times V_2) \quad = \quad (U_1 \cap U_2) \times (V_1 \cap V_2).$$

Since $(x,y) \in U_1 \times V_1$ and $(x,y) \in U_2 \times V_2$, $(x,y) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$. Therefore, take $U_3 \times V_3 = (U_1 \cap U_2) \times (V_1 \cap V_2) \in \mathcal{N}(x,y)$ and $U_3 \times V_3 \subseteq (U_1 \cap U_2) \times (V_1 \cap V_2)$.

(iii') Take $U_1 \times V_1 \in \mathcal{N}(x,y)$ with $(x_1, y_1) \in U_1 \times V_1$. It will suffice to find a $U_2 \times V_2 \in \mathcal{N}(x_1, y_1)$ that is a subset of $U_1 \times V_1$. Such a set is $U_2 \times V_2 = U_1 \times V_1$. By assumption $(x_1, y_1) \in U_1 \times V_1$, this implies $U_2 \times V_2 \in \mathcal{N}(x_1, y_1)$ with $U_2 \times V_2 \subseteq U_1 \times V_1$.

$\therefore$ $\mathcal{N}(x,y)$ is a local base.

4. $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2)$ are topological spaces. The topology $\mathcal{T}_1$ is said to be *finer* than $\mathcal{T}_2$ if $\mathcal{T}_2 \subseteq \mathcal{T}_1$.

(a) Show that $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ iff for every $x \in X$ and $(U \ni x) \in \mathcal{T}_2$, there is a $(V \ni x) \in \mathcal{T}_1$ such that $V \subseteq U$.

Solution. ($\implies$) Assume $\mathcal{T}_1$ is finer than $\mathcal{T}_2$, which implies $\mathcal{T}_2 \subseteq \mathcal{T}_1$ by definition. Therefore, all open sets in $\mathcal{T}_2$ are also in $\mathcal{T}_1$. Pick some open set, $U \in \mathcal{T}_2$, which contains x. Clearly $U \in \mathcal{T}_1$, which is sufficient to prove the forward direction with $V = U \subseteq U$, which contains x by construction.

Therefore, $U \in \mathcal{T}_2 \subseteq \mathcal{T}_1 \implies$ there is a $(V \ni x) \in \mathcal{T}_1$ such that $V \subseteq U$.

($\impliedby$) Assume $\forall x \in X$, $(U \ni X) \in \mathcal{T}_2$, there is a $(V \ni x) \in \mathcal{T}_1$ s.t. $V \subseteq U$. Said in a different way, for all open sets $U_\alpha \in \mathcal{T}_2$, $\exists V_\alpha \subseteq U_\alpha$ in $\mathcal{T}_1$. No matter what set I pick that contains x in $\mathcal{T}_2$, I can find a set (neighborhood) in $\mathcal{T}_1$ that is contained within the first set or is the first set. Therefore, all sets in $\mathcal{T}_2$ must be contained within $\mathcal{T}_1$, which implies $\mathcal{T}_2 \subset \mathcal{T}_1$. Now, by assumption, since V may be equal U, we get that the containment may not be strict, that is $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Therefore, $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

Therefore, $\forall x \in X$, $(U \ni X) \in \mathcal{T}_2$, there is a $(V \ni x) \in \mathcal{T}_1$ s.t. $V \subseteq U \implies \mathcal{T}_1$ is finer than $\mathcal{T}_2$.

$\square$ $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ iff for every $x \in X$ and $(U \ni x) \in \mathcal{T}_2$, there is a $(V \ni x) \in \mathcal{T}_1$ such that $V \subseteq U$.

(b) Show that the l^1 and l^2 metrics on $\mathbb{R}^2$ generate the same topology.

Solution. Let $\mathcal{T}_1$ be the topology generated by the metric space $(\mathbb{R}^2, l^1)$ and $\mathcal{T}_2$ be the topology generated by the metric space $(\mathbb{R}^2, l^2)$. $B_1(x,\epsilon) = \{y \in \mathbb{R}^2 : ||x-y||_1 < \epsilon\}$ and $B_2(x,\epsilon) = \{y \in \mathbb{R}^2 : ||x-y||_2 < \epsilon\}$ are ϵ-balls about $x \in \mathbb{R}^2$ in $(\mathbb{R}^2, l^1)$ and $(\mathbb{R}^2, l^2)$ respectively.

Suppose $U \in \mathcal{T}_1$. Then for every $x \in U$ there is an ϵ such that $B_1(x,\epsilon) \subseteq U$. The l^1 and l^2 metrics are equivalent on $\mathbb{R}^2$, so there exists $c_1 > 0$ s.t. $||x-y||_1 \le c_1||x-y||_2$ for all $y \in U$. It follows that $B_2(x, \frac{\epsilon}{c_1}) \subseteq B_1(x,\epsilon) \subseteq U$ since $||x-y||_2 < \frac{\epsilon}{c_1}$ implies $||x-y||_1 < \epsilon$. So, for every $x \in U$ there is an $\epsilon' = \frac{\epsilon}{c_1}$ such that $B_2(x,\epsilon') \subseteq U$. Thus, $U \in \mathcal{T}_2$.

Suppose $U \in \mathcal{T}_2$. Then for every $x \in U$ there is an ϵ such that $B_2(x,\epsilon) \subseteq U$. The l^1 and l^2 metrics are equivalent on $\mathbb{R}^2$, so there exists $c_2 > 0$ s.t. $||x-y||_2 \le c_2||x-y||_1$ for all $y \in U$. It follows that $B_1(x, \frac{\epsilon}{c_2}) \subseteq B_2(x,\epsilon) \subseteq U$ since $||x-y||_1 < \frac{\epsilon}{c_2}$ implies $||x-y||_2 < \epsilon$. So, for every $x \in U$ there is an $\epsilon' = \frac{\epsilon}{c_2}$ such that $B_1(x,\epsilon') \subseteq U$. Thus, $U \in \mathcal{T}_1$.

$\square$ the l^1 and l^2 metrics on $\mathbb{R}^2$ generate the same topology.

(c) Let $X = l^1(\mathbb{R}, \mathbb{N})$. Find a set $A \subseteq X$ that is open wrt the l^1 metric topology on X, but not open wrt the l^∞ metric topology on X.

Solution. First let $B_1(x,r)$ be the ball centered at x with radius r such that $B_1(x,r) = \{y|\ ||x-y||_1 < r\}$. Similarly, define $B_\infty(x,r)$. Next let

$$A = B_1(0,1) = \{x| \sum_{i=0} |x_i| < 1\}.$$

A is clearly open since, for all $x \in B_1(0,1)$ we can find an $\epsilon > 0$ such that $B_1(x,\epsilon) \subset B_1(0,1)$. Now consider $x = 0 = (0,0,\dots,0,\dots)$. Note that $x \in B_1(0,1)$ and $x \in B_\infty(0,1)$. For all $\epsilon > 0$ define $y = (\epsilon/2, \epsilon/2, \dots, \epsilon/2, \dots)$. Now $||x-y||_\infty = \epsilon/2$ and thus $y \in B_\infty(0,\epsilon)$, but $||y||_2 \to \infty$ and thus $y \notin A$.

$\therefore$ A is not open in l^∞.

(d) Let $X = C([0,1], \mathbb{R})$ and let d_1 and d_2 be the L^1 and L^2 metrics respectively. Show that the metric topology generated by the L^2 metric is strictly finer than the metric given by the L^1 topology.

Solution. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the metric topologies generated by the L^1 and L^2 metrics respectively. $B_1(f,\epsilon) = \{g \in C([0,1],\mathbb{R}) : ||f-g||_{L^1} < \epsilon\}$ and $B_2(f,\epsilon) = \{g \in C([0,1],\mathbb{R}) : ||f-g||_{L^2} < \epsilon\}$ are ϵ-balls about $f \in C([0,1],\mathbb{R})$ in the L^1 and L^2 metric topologies respectively.

Suppose $U \in \mathcal{T}_1$. Then for every $f \in U$ there is an ϵ such that $B_1(f,\epsilon) \subseteq U$. By Hölder's inequality, we know that $||f-g||_1 \le ||f-g||_2$ for all $g \in U$. It follows that $B_2(f,\epsilon) \subseteq B_1(f,\epsilon) \subseteq U$ since $||f-g||_2 < \epsilon$ implies $||f-g||_2 < \epsilon$. So, for every $f \in U$ there is an $\epsilon' = \epsilon$ such that $B_2(f,\epsilon') \subseteq U$. Thus, $U \in \mathcal{T}_2$ and $\mathcal{T}_1 \subseteq \mathcal{T}_2$. Consider the unit ball in the L^2 metric topology centered at the zero function. $A = B_2(0,1) \subseteq X$. Clearly,

A is open wrt the L^2 metric topology centered at the zero sequence has $f \in B_1(0, \epsilon)$ where:

$$f(x) \quad = \quad \begin{cases} \frac{3}{\epsilon} - \frac{9}{\epsilon^3}x & 0 \leq x \leq \frac{\epsilon^2}{3} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} ||f||_{L^1} &= \int_0^1 |f(x)| dx \\ &= \int_0^{\frac{\epsilon^2}{3}} \left(\frac{3}{\epsilon} - \frac{9}{\epsilon^3}x \right) dx \\ &= \left[\frac{3}{\epsilon}x - \frac{9}{2\epsilon^3}x^2 \right]_{x=0}^{\frac{\epsilon^2}{3}} \\ &= \frac{\epsilon}{2} \end{aligned}$$

$$\begin{aligned} ||f||_{L^2}^2 &= \int_0^1 |f(x)|^2 dx \\ &= \int_0^{\frac{\epsilon^2}{3}} \left(\frac{3}{\epsilon} - \frac{9}{\epsilon^3}x \right)^2 dx \\ &= \left[-\frac{\epsilon^3}{27} \left(\frac{3}{\epsilon} - \frac{9}{\epsilon^3}x \right)^3 \right]_{x=0}^{\frac{\epsilon^2}{3}} \\ &= 1 \end{aligned}$$

So $f \notin A$, which implies $B_1(0, \epsilon) \subseteq A$ for no $\epsilon > 0$. Thus, A is not open wrt the L^1 metric topology on X.